International Journal of Innovative Computing, Information and Control
Volume **5**, Number **1**, January **2009**
ICIC International ©2009 ISSN 1349-4198
pp. **17–27**

ON PARAMETER ESTIMATION OF STOCHASTIC VOLATILITY MODELS FROM STOCK DATA USING PARTICLE FILTER -APPLICATION TO AEX INDEX-

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Received February 2008; revised June 2008

Abstract. *We consider the problem of estimating stochastic volatility from stock data. The estimation of the volatility process of the Heston model is not in the usual framework of the filtering theory. Discretizing the continuous Heston model to the discrete-time one, we can derive the exact volatility filter and realize this filter with the aid of particle filter algorithm. In this paper, we derive the optimal importance function and construct the particle filter algorithm for the discrete-time Heston model. The parameters contained in system model are also estimated by constructing the augmented states for the system and parameters. The developed method is applied to the real data (AEX index).*
Keywords: Stochastic volatility, Heston model, Parameter estimation, Particle filter, AEX index

1. **Introduction.** Due to the apperant contradiction of constant volatility assumption of the Black-Sholes model as illustrated by the volatility skew observed in practice, the stochastic volatility models were proposed and applied to the option pricing problems in [1, 2]. We consider the simple stochastic volatility model proposed by Heston [2]:

$$dS_t = \mu_S S_t dt + \sqrt{v_t} S_t dB_t \tag{1}$$
$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t} dZ_t \tag{2}$$

where B_t and Z_t are standard Brownian motion processes with the correlation ρ. From the observed stock price S_t, we construct the transformed observation process $y_t = \log S_t/S_0$;

$$dy_t = (\mu_S - \frac{1}{2}v_t)dt + \sqrt{v_t} dB_t. \tag{3}$$

We are interested in estimating the volatility process v_t, for each fixed t, based on our observation data $\{y_s\}_{0\le s\le t}$. Setting

$$\tilde{Z}_t = \frac{1}{\sqrt{1-\rho^2}}(Z_t - \rho B_t),$$

we find that $\tilde{Z}_t$ is independent of B_t. Noting that

$$\begin{aligned} dZ_t &= \sqrt{1-\rho^2} d\tilde{Z}_t + \rho dB_t \\ &= \sqrt{1-\rho^2} d\tilde{Z}_t + \frac{\rho}{\sqrt{v_t}}(dy_t - (\mu_S - \frac{1}{2}v_t)dt), \end{aligned}$$

we have

$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}\sqrt{1-\rho^2}d\tilde{Z}_t + \xi\rho(dy_t - (\mu_S - \frac{1}{2}v_t)dt). \quad (4)$$

Our problem then is to find the "best" estimate of v_t given by (4) based on the observation $\{y_s\}_{0\leq s\leq t}$, where y_t evolves according to (3). This is the usual filtering problem as it appears in signal processing and stochastic control theory. However, the standard filtering theory algorithm can not be applied in this situation ([3],[4]). The reason for this is clearly explained in [5] and [6].

Recent results for the filtering of the stochastic volatility in the continuous time frame work can be found in [7]. As stated in [7], we need to solve the Zakai equation to obtain the stochastic volatility estimate. Although the robust form of the Zakai equation can be derived with the splitting up method, the obtained results are very sensitive to the noise correlation parameter. The numerical behavior is very complicated and does not always work well.

To circumvent the above difficulty, we envisage here the use of particle filter for volatility estimation as proposed in [8]. Particle filter is a simulation based tool for filtering in discrete time framework, which can easily adapt to the nonlinearity in the model and/or non Gaussian noises. Here, the probability distributions are represented by a cloud of (weighted) particles. These particles are recursively generated from a so called "importance function", $\pi(\cdot)$. Although the resulting densities (represented by the particle clouds) do asymptotically converge to the true filtered densities as the number of particles tends to infinity, the efficiency of this method depends heavily on the importance function used. Usually the 'naive' proposal, $p(v_k|v_{k-1})$, which is the discrete state transition density, is used as the importance function due to the ease of drawing samples from it and corresponding simplicity of weight update [9]. However, a better choice for importance function is, $\pi = p(v_k|v_{k-1}^i, y_k)$, i.e. to make use of recent observation as it carries information about the state v_k. Moreover, as shown by [10], this is also optimal in the sense that the variance of the importance weights is minimum.

In this paper, we actually evaluate the optimal importance function after discretizing the Heston model. We then implement the particle filter using this optimal importance function. We then address the parameter estimation problem by augmenting the state and parameters. We propose here a new algorithm where the parameter is estimated by weighted average of some set of particles selected initially from any arbitrary (possibly uniform) distribution. The weights chosen come from the weight updates for the state "particles". We obtain the feasible parameter estimates without adding extra noise. The feasibility of the proposed method is first demonstrated using simulated data and next, we apply this to the AEX index data.

2. **Particle Filter and Optimal Importance Function.**

Here we present the particle filter formulation and the selection of the optimal importance function.

In order to apply the particle filter to our system, we discretize the system (4) and (3) using Euler scheme. We select this scheme mainly due to its relative simplicity and less computational load.The discretized system is given as

$$\begin{aligned} v_k &= v_{k-1} + \kappa(\theta - v_{k-1})\Delta t - \xi\rho(\mu_S - \frac{1}{2}v_{k-1})\Delta t \\ &\quad + \xi\sqrt{v_{k-1}}\sqrt{1-\rho^2}\Delta\tilde{Z}_k + \xi\rho(y_k - y_{k-1}) \end{aligned} \quad (5)$$

and

$$y_k = y_{k-1} + (\mu_S - \frac{1}{2}v_k)\Delta t + \sqrt{v_{k-1}}\Delta B_k$$

where for $t_k - t_{k-1} = \Delta t$,

$$\Delta B_k = B_{t_k} - B_{t_{k-1}},\ \Delta\tilde{Z}_k = \tilde{Z}_{t_k} - \tilde{Z}_{t_{k-1}}.$$

Now we use the sequential importance sampling (with resampling) algorithm for the particle filter.

- The updated weight $w_k^{(i)}$ at t_k is obtained by
$$w_k^{(i)} = w_{k-1}^{(i)} \frac{p(y_k|v_{0:k}^{(i)}, y_{0:k-1})p(v_k^{(i)}|v_{0:k-1}^{(i)}, y_{0:k-1})}{\pi(v_k^{(i)}|v_{k-1}^{(i)}, y_k)}.$$
- It is possible to select the importance function π as the optimal selection in [10]
$$\pi(v_k|v_{k-1}, y_k) = p(v_k|v_{k-1}, y_k).$$
- Form (5) we can easily get
$$p(v_k|v_{k-1}, y_k) = \mathcal{N}(m(v_{k-1}, y_k), \sigma^2(v_{k-1}))$$
where
$$\begin{aligned} m(v_{k-1}, y_k) &= v_{k-1} + \kappa(\theta - v_{k-1})\Delta t \\ &\quad - \xi\rho(\mu_S - \frac{1}{2}v_{k-1})\Delta t + \xi\rho(y_k - y_{k-1}) \end{aligned}$$
and
$$\sigma(v_{k-1}) = \xi\sqrt{v_{k-1}^{(i)}}\sqrt{1-\rho^2}\sqrt{\Delta t}.$$

Hence we obtain the optimal importance function for the particle filter. There is always a small probability of getting negative value during sampling. For simplicity, we discard those samples and replace them with new positive samples.

- Next problem is to obtain the $p(v_k|v_{0:k-1}, y_{0:k-1})$ -function: Now substituting the observation data y_k into (5), we obtain
$$\begin{aligned} v_k &= v_{k-1} + \kappa(\theta - v_{k-1})\Delta t - \xi\rho(\mu_S - \frac{1}{2}v_{k-1})\Delta t + \xi\sqrt{v_{k-1}}\sqrt{1-\rho^2}\Delta\tilde{Z}_k \\ &\quad + \xi\rho\{(\mu_S - \frac{1}{2}v_k)\Delta t + \sqrt{v_{k-1}}\Delta B_k\} \end{aligned}$$
i.e.,
$$\begin{aligned} v_k &= (1 + \frac{1}{2}\xi\rho\Delta t)^{-1}\{v_{k-1} + \kappa(\theta - v_{k-1})\Delta t \\ &\quad + \xi\rho\frac{1}{2}v_{k-1}\Delta t + \xi\sqrt{v_{k-1}}\sqrt{1-\rho^2}\Delta\tilde{Z}_k + \xi\rho\sqrt{v_{k-1}}\Delta B_k\}. \end{aligned}$$
This implies that
$$p(v_k|v_{0:k-1}, y_{0:k-1}) = p(v_k|v_{k-1}) = \mathcal{N}(\tilde{m}(v_{k-1}), \tilde{\sigma}^2(v_{k-1}))$$
where
$$\tilde{m}(v_{k-1}) = (1 + \frac{1}{2}\xi\rho\Delta t)^{-1}\{v_{k-1} + \kappa(\theta - v_{k-1})\Delta t + \frac{\xi\rho}{2}v_{k-1}\Delta t$$
and
$$\tilde{\sigma}(v_{k-1}) = (1 + \frac{1}{2}\xi\rho\Delta t)^{-1}\xi\sqrt{v_{k-1}}\sqrt{\Delta t}.$$
- The likelihood function becomes $p(y_k|v_{0:k}, y_{0:k-1}) = p(y_k|v_k, v_{k-1}, y_{k-1})$, which is given as
$$p(y_k|v_k, v_{k-1}, y_{k-1}) = \mathcal{N}(y_{k-1} + (\mu - \frac{1}{2}v_k\Delta t, v_{k-1}\Delta t).$$

3. **Parameter Identification Problem.** To identify the parameters contained in the system model, we construct the augmented state $z_k = (v_k, \alpha)$ where vector α contains the parameters as

$$\alpha = [\kappa \ (\kappa\theta) \ \xi \ \mu_S \ \rho].$$

To perform the particle filter for z_k we assume that

$$\alpha \in U(uniform\ distribution\ with\ known\ upper\ and\ lower\ bounds), \tag{6}$$

and is independent of the initial distribution of $v_1 \in \mathcal{N}$. Hence we can apply the particle filter algorithm developed in the previous section to z_k-process. Noting that the state α is time independent, the parameter value $\alpha^{(i)}$ is not updated and we encounter the so called degeneration problem. In this paper to avoid this deficiency, we use the simple random resampling for each parameter and apply the systematic resampling for the state v_k.

4. **Simulation Studies.** In the following simulations, resampling is done whenever the effective sample size as defined in [10] falls below two-third of the sample size used.

First we check the algorithm developed here. Setting the parameters:

$$\kappa = 3.0,\ \theta = 0.1,\ \mu = 0.1,\ \rho = -0.2,\ \xi = 0.4,$$

the stock price and volatility processes are synthetically generated. The simulated volatility and the log price $y(t)$ are shown in Figure 1 and Figure 2 respectively. In the simulation studies, we set $\Delta t = 0.001$ and the number of particles is set as 2000 for each augmented state. For unknown parameters, we set

$$\kappa \in U[1,9] \quad \kappa\theta \in U[0.05, 0.4] \quad \mu \in U[0.05, 0.5]$$
$$\xi \in U[0.01, 0.91] \ \rho \in U[-0.5, 0]$$

and

$$v_1 \in \mathcal{N}(0.25, 0.02^2).$$

The true and estimated volatility are shown in Figure 3. The square error as defined by $|v_k - \widehat{v}_k|^2$ is shown in Figure 4.

The estimates of unknown parameters are also demonstrated in Figures 5 to 9.

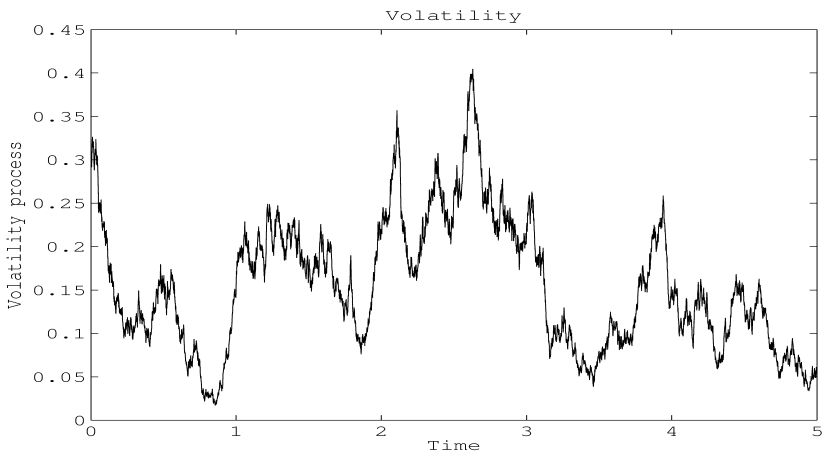

FIGURE 1. Simulated volatility

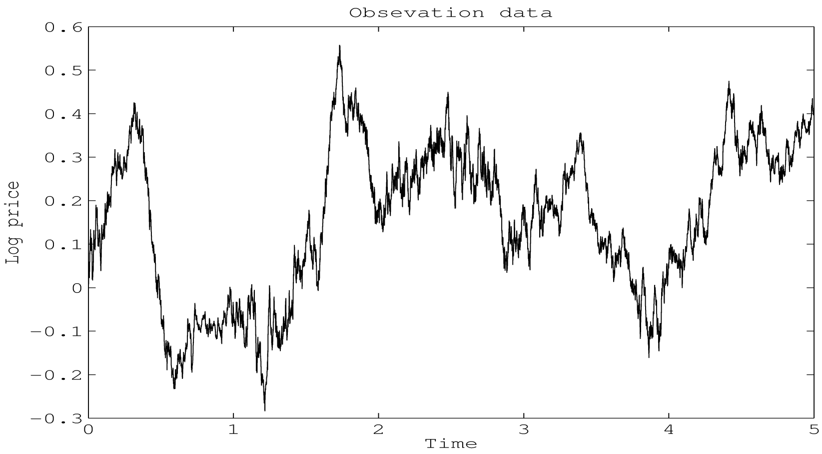

FIGURE 2. Observation data (log price)

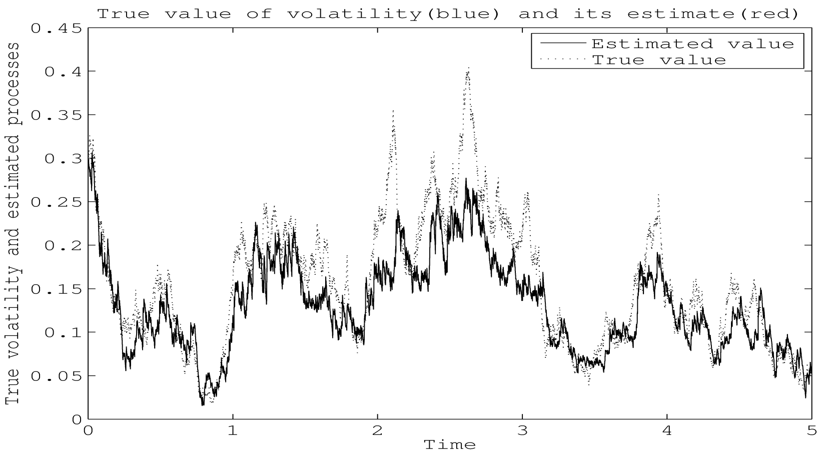

FIGURE 3. True and estimated volatility processes

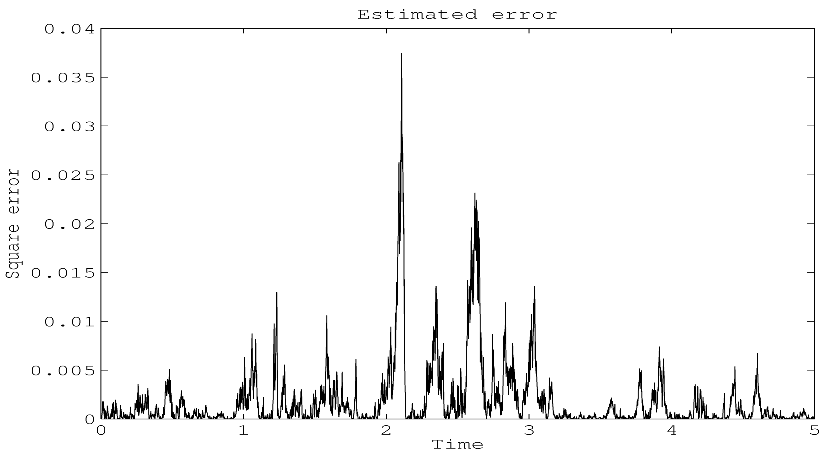

FIGURE 4. Square error of estimate v

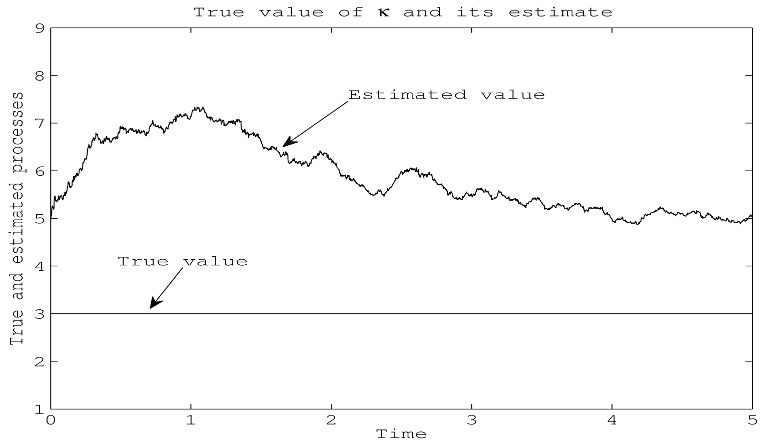

FIGURE 5. True and estimated κ

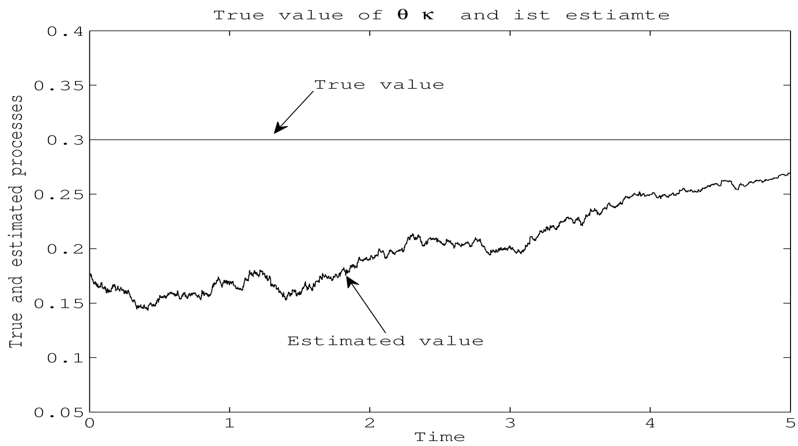

FIGURE 6. True and estimated θ

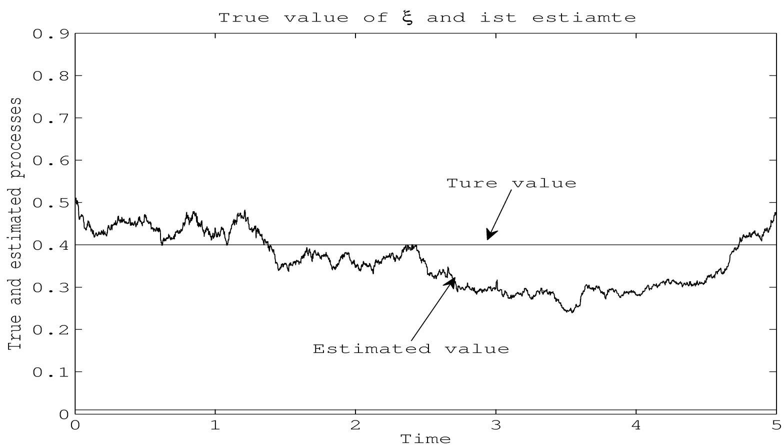

FIGURE 7. True and estimated ξ

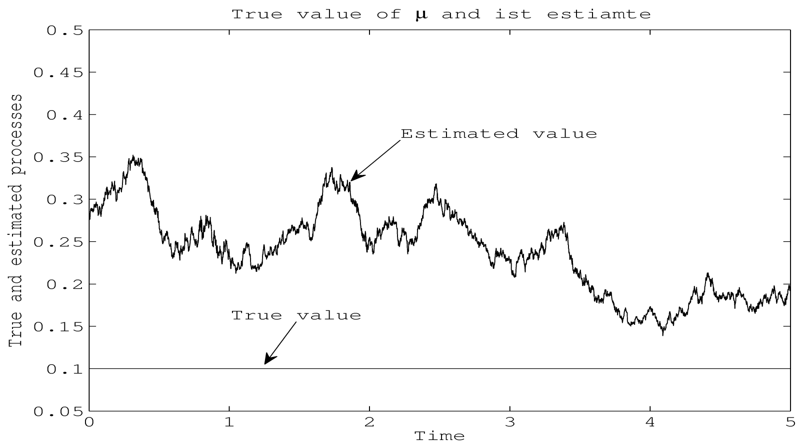

FIGURE 8. True and estimated μ

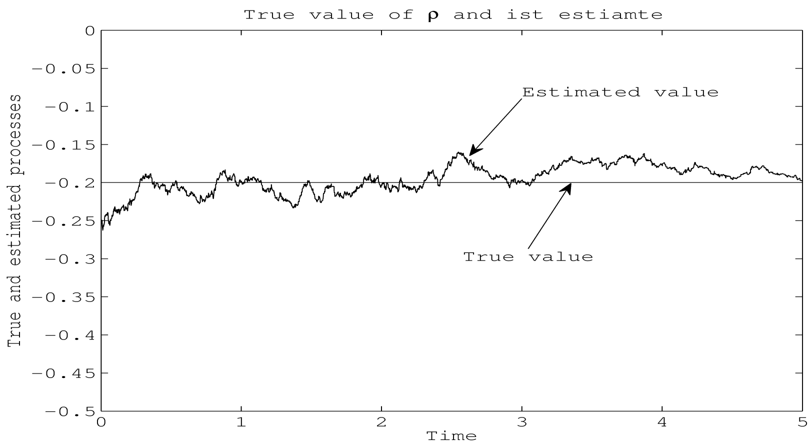

FIGURE 9. True and estimated ρ

5. **Application to AEX Index Data.** In this section, we apply the method developed in Section 2 to the AEX index data. The AEX index is a stock market index composed of Dutch companies that trade on Euronext Amsterdam. This index is composed of a maximum of 25 of the most actively traded securities on the exchange.

Starting from 03 Jan. 2000, we observe the AEX index on each day as shown in Figure 10. Its log price is also shown in Figure 11.

We set the time difference as $\Delta t = 1/252$(year) and the number of particles is set as 2000 for each augmented state. For unknown parameters, we set

$$\kappa \in U[1,10] \quad \kappa\theta \in U[0.1,4.5] \quad \mu \in U[-0.2,0.3]$$
$$\xi \in U[0.1,0.6] \; \rho \in U[-0.8,-0.1]$$

We also set

$$v_1 \in \mathcal{N}(0.27, 0.02^2).$$

The estimated results are shown from Figure 12 to Figure 16.

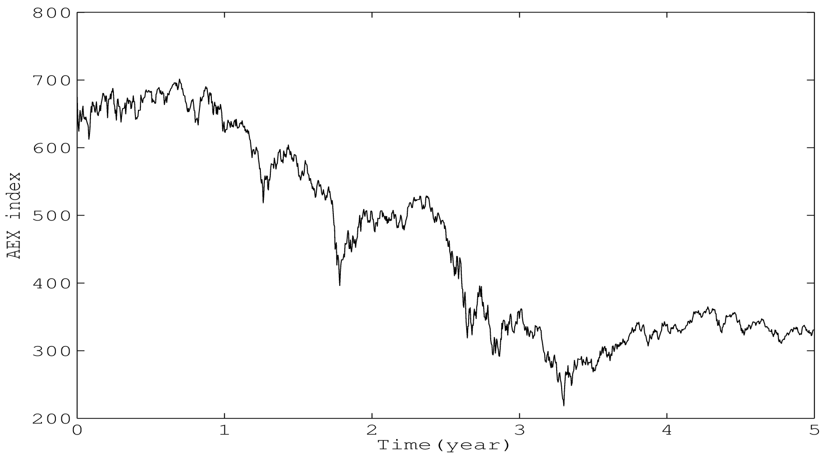

FIGURE 10. AEX index data

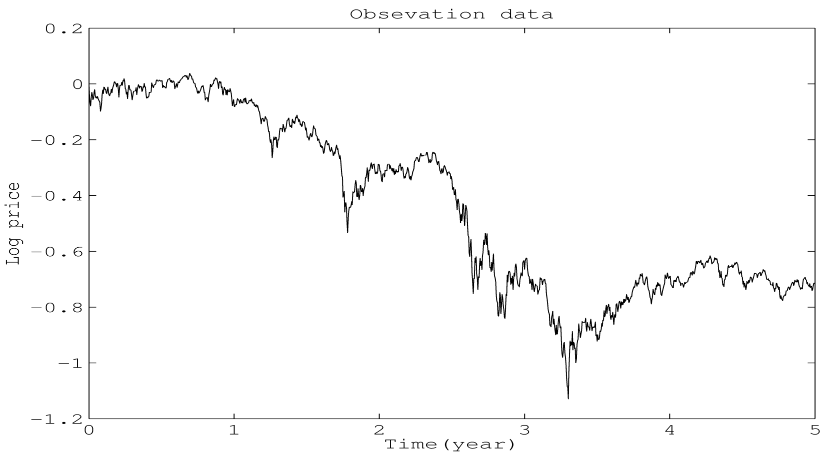

FIGURE 11. log price

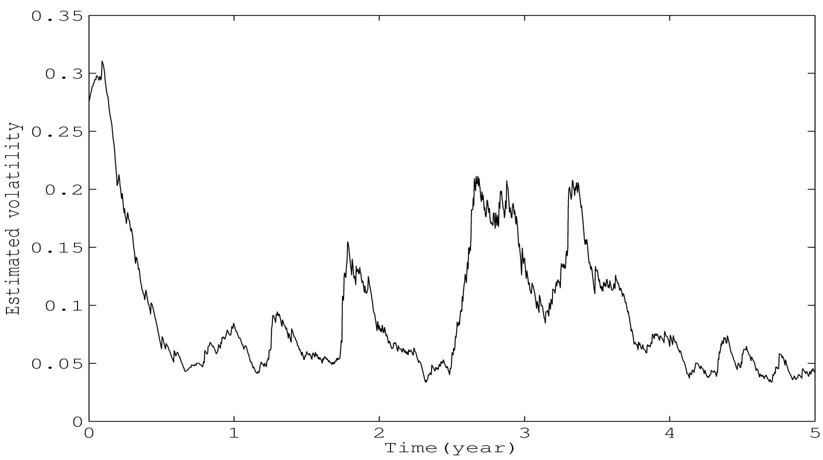

FIGURE 12. Estimated volatility

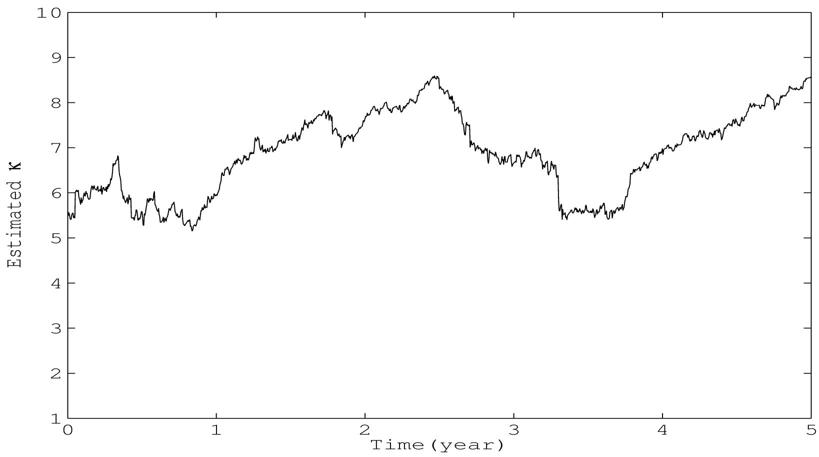

FIGURE 13. Estimated κ

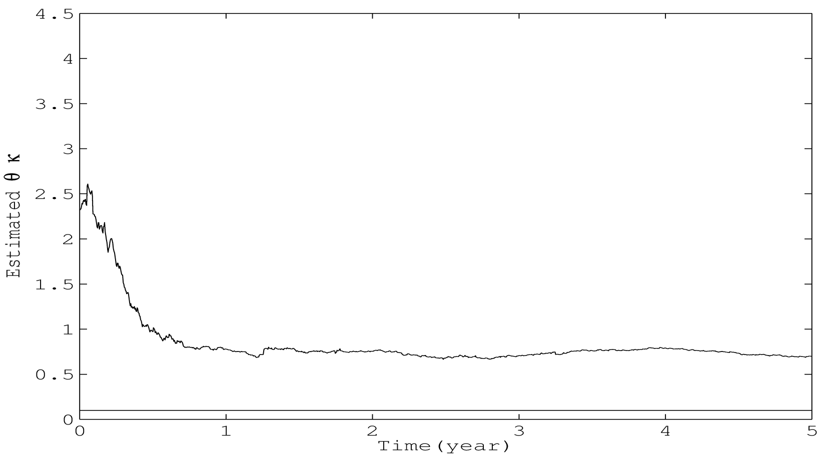

FIGURE 14. Estimated $\kappa\theta$

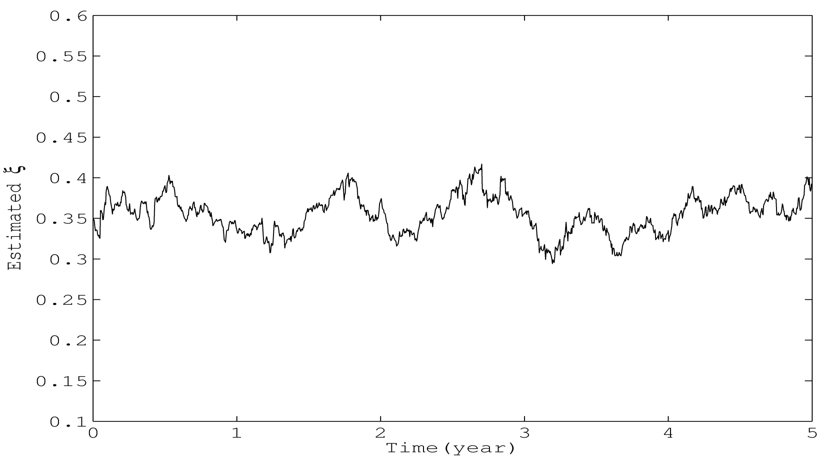

FIGURE 15. Estimated ξ

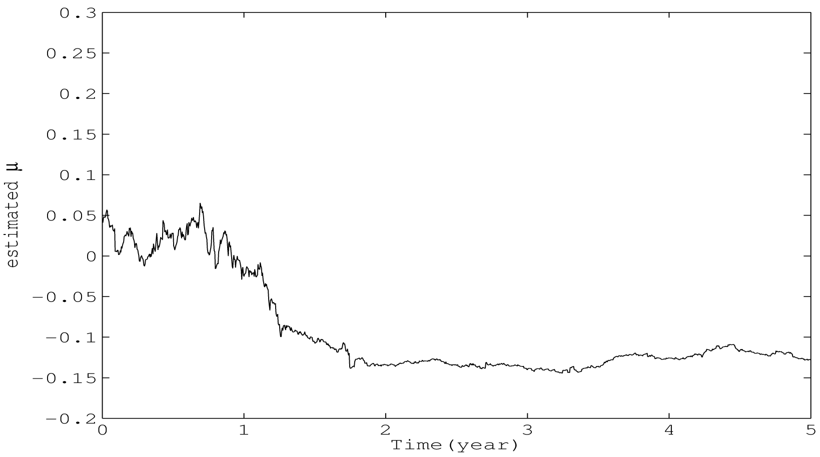

FIGURE 16. Estimated μ

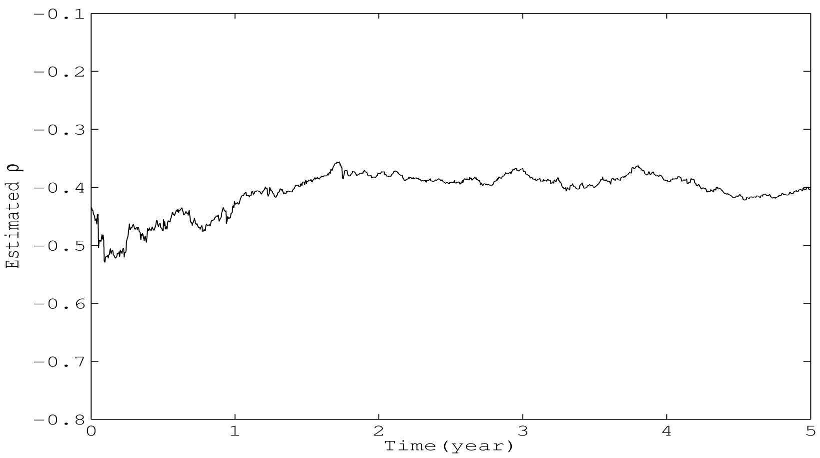

FIGURE 17. Estimated ρ

For the real stock data, the exact values of the underlying stochastic volatility and the model parameters are unknown. Hence it is difficult to compare the estimates obtained. However, from the results of the previous numerical experiment using simulated data, we have observed that with the estimated model parameters, the estimation of stochastic volatility works quite well. So, one can expect that the corresponding estimates using the real stock data to be quite reasonable.

6. **Conclusion.** For the discretized Heston model, we estimate the stochastic volatility using particle filter with the optimal importance function. Using the simple random resampling method, we also propose an estimation procedure for the parameters of the model. The algorithms developed have been applied to the AEX index data for estimating the stochastic volatility together with the model parameters.

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